Chapter 8:

Sequences and summation

Definitions

- Sequence: an ordered list of elements
  - Like a set, but:
    - Elements can be duplicated
    - Elements are ordered
- A sequence is a function from a subset of **Z** (integer numbers) to a set S
  - a_n is the image of n
  - a_n is a term in the sequence
  - $\{a_n\}$ means the entire sequence

Sequence examples

- $a_n = 3n$
  - The terms in the sequence are $a_1, a_2, a_3, \ldots$
  - The sequence $\{a_n\}$ is $\{ 3, 6, 9, 12, \ldots \}$
- $b_n = 2^n$
  - The terms in the sequence are $b_1, b_2, b_3, \ldots$
  - The sequence $\{b_n\}$ is $\{ 2, 4, 8, 16, 32, \ldots \}$
- Note that sequences are indexed from 1
  - Not in all other textbooks, though!

Geometric vs. arithmetic sequences

- The difference is in how they grow
- Arithmetic sequences increase by a constant *amount*
  - it is a sequence of form a, a+d, a+2d,………, a + nd

where the *initial term* **a** and the *common difference* **d** are real numbers

Ex: $a_n = 3n$

- The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }
- Each number is 3 more than the last
- initial term **a** =3
- common difference **d** = 3

Geometric vs. arithmetic sequences

- Geometric sequences increase by a constant *factor*
- it is a sequence of form a, ar, ar^2, ar^3 ………,ar^n

where the <u>*initial term*</u> is **a** and the <u>*common ratio*</u> **r** are real numbers.

EX: $b_n = 2^n$

- The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }
- Each number is twice the previous
- initial term **a** =2
- common ratio **r** = 32

Geometric vs. arithmetic sequences

- Ex: the sequence $\{b_n\}$ with $b_n = (-1)^n$, $\{C_n\}$with $C_n = 2*5^n$ are geometric progression
  - b_n = -1, 1, -1, 1,……..

    initial term = -1,

    common ratio= -1
  - C_n = 10,50,250,1250,……….

    initial term = 10,

    common ratio= 5

Geometric vs. arithmetic sequences

- Ex: the sequence $\{S_n\}$ with $Sn = -1 + 4n$ is arithmetic sequence where

$S_n = 3, 7, 11, \ldots\ldots\ldots$

initial term = 3,

common Difference= 4

Special, Integer Sequences

- Ex: Find a formula for the following sequences:

A. 1, ½, ¼, 1/8 , 1/16

Sol: $a_n = 1/ 2^{n-1}$ it a geometric progression with initial term = 1 and common ration= ½

B. 5, 11, 17, 23, 29 …

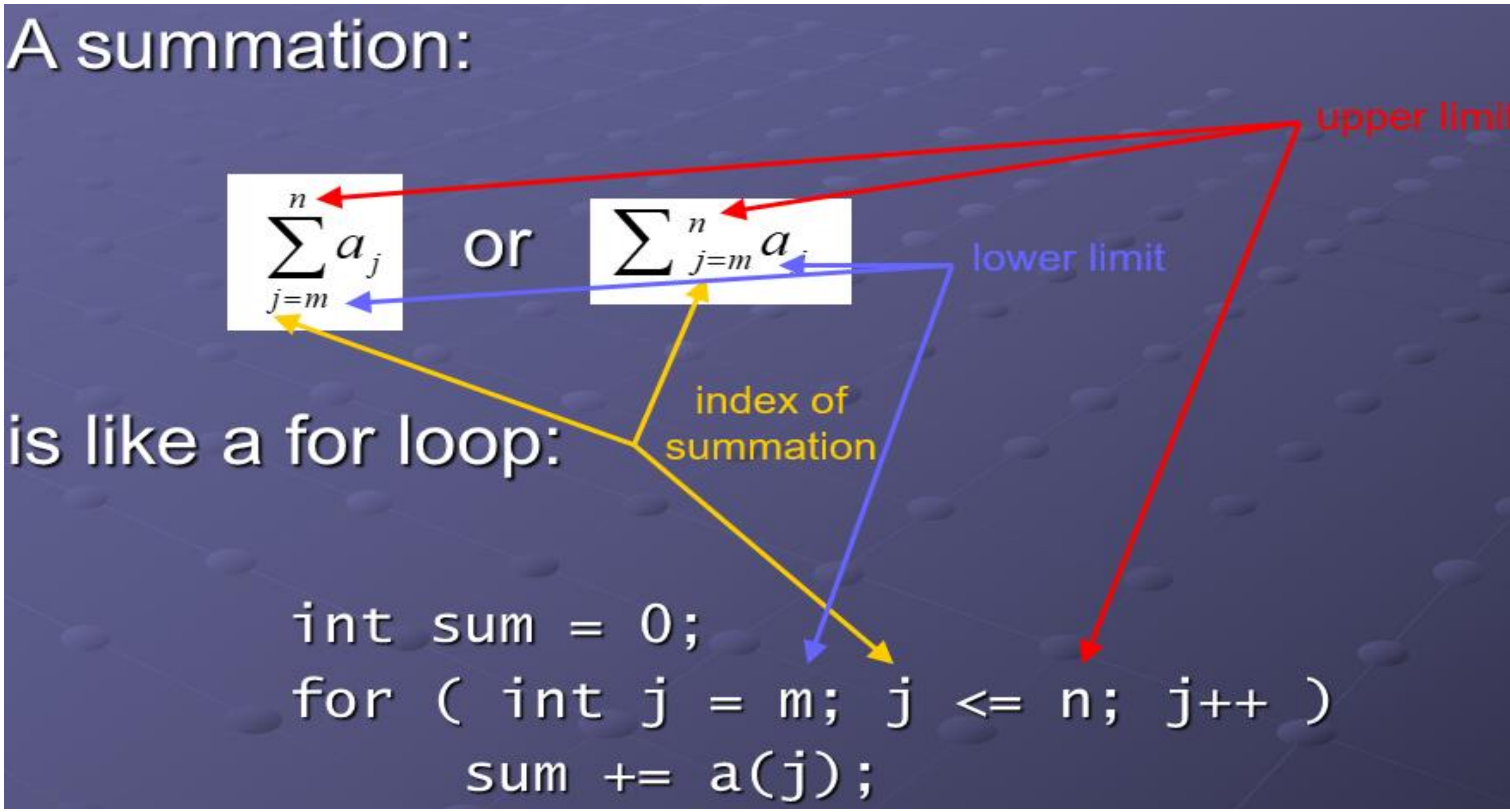
Sol: $a_n = 6n – 1$ is arithmetic progression with a= 5, and d= 6

Useful sequences

Nth term	First 5 terms
n^2	1, 4, 9, 16, 25,……..
2^n	2, 4, 8, 16, 32,………
n!	1, 2, 6, 24, 120, …….

Note: to know more useful sequences go to page 228 table 1

Summations


A summation:
$\sum_{j=m}^{n} a_j$
or
$\sum_{j=m}^{n} a_j$
upper limit
lower limit
index of summation
is like a for loop:
int sum = 0;
for (int j = m; j <= n; j++)
sum += a(j);

Evaluating sequences

$$\sum_{k=1}^{5}(k+1) = 2+3+4+5+6 = 20$$

$$\sum_{j=0}^{4}(-2)^j = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$$

$$\sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 = 30$$

$$\sum_{i=1}^{n} c = c*n \Longrightarrow \sum_{i=1}^{10} 3 = 10*3 = 30$$

$$\sum_{j=0}^{8}\left(2^{j+1} - 2^{j}\right) = (2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + \ldots (2^9 - 2^8) = 511$$

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4}(K+1)^2 = \sum_{L=2}^{6}(L-1)^2 = 1+4+9+16+25 = 55$$

Double summations

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3)$$

$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4) = 6*10 = 60$$

Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
  for ( int j = 1; j <= 3; j++ )
       sum += i*j;
```

Useful summation formulae

$$\sum_{k=0}^{n} ar^k = a(r^{n+1} - 1)/(r-1), r \neq 1$$

$$\sum_{k=1}^{n} k = n(n+1)/2$$

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$$

$$\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$$

$$\sum_{K=o}^{\infty} x^k = 1/(1-x), |x| < 1$$

$$\sum_{K=0}^{\infty} kx^{k-1} = 1/(1-x)2, |x| < 1$$

Evaluating sequences

Ex: find the value of

$$\sum_{k=50}^{100} k^2$$

$$\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$$

$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338{,}350 - 40{,}425$$

$$= 297{,}925.$$

Evaluating sequences

Ex: find the value of

$$\sum_{k=2}^{100} k^2$$

$$\sum_{k=2}^{100} k^2 = \sum_{k=1}^{100} k^2 - (1)^2$$

Evaluating sequences

Ex: find the value of

$$\sum_{k=-2}^{100} k^2$$

$$\sum_{k=-2}^{100} k^2 = \sum_{k=1}^{100} k^2 + (-2)^2 + (-1)^2 + (0)^2$$

Evaluating sequences

$$\text{Ex: Given that } \sum_{k=1}^{100} k^2 = 338{,}350 \text{ find the value of :}$$

$$A. \sum_{k=1}^{99} k^2 \qquad B. \sum_{k=1}^{101} k^2$$

$$A. \sum_{k=1}^{99} k^2 = \sum_{k=1}^{100} k^2 - (100)^2 = 338{,}350 - 10{,}000 = 328{,}350$$

$$B. \sum_{k=1}^{101} k^2 = \sum_{k=1}^{100} k^2 + (101)^2 = 338{,}350 + 10{,}201 = 348{,}551$$

$$\text{Ex: Given that } \sum_{k=1}^{100} k^2 = 338{,}350 \text{ find the value of :} \quad \sum_{k=1}^{100} k^2 + 10 =$$

$$\sum_{k=1}^{100} k^2 + 10 = \sum_{k=1}^{100} 10 + \sum_{k=1}^{100} k^2 = 100 * 10 + 338{,}350 = 339{,}350$$